ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca=2$. Prove that :

$$\frac{a+b+c}{3}\geq\sqrt{\frac{a^2b+b^2c+c^2a+2}{6}}$$

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By AM − GM inequality, we have

$$4ab^2+a(2-c)^2\geq 4ab(2-c)\ \ \text{(and analogs)}$$

Adding this inequality with the similar ones, we get

$$4(a+b+c)+4\left(ab^2+bc^2+ca^2\right)+\left(a^2b+b^2c+c^2a\right)\geq 24-12abc$$

$$\Leftrightarrow 4(a+b+c)+4(a+b+c)(ab+bc+ca)\geq 3\left(a^2b+b^2c+c^2a\right)+24$$

$$\Leftrightarrow 12(a+b+c)\geq 3\left(a^2b+b^2c+c^2a\right)+24$$

$$\Rightarrow 2(a+b+c)^2+18\overset{AM-GM}{\geq}12(a+b+c)\geq 3\left(a^2b+b^2c+c^2a\right)+24$$

$$\Rightarrow\frac{a+b+c}{3}\geq\sqrt{\frac{a^2b+b^2c+c^2a+2}{6}}.$$

Equality holds iff $(a,b,c)=(2,1,0)$ and its cyclic permutations.