ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca=1$. Prove that :

$$\frac{a}{\sqrt{a+1}}+\frac{b}{\sqrt{b+1}}+\frac{c}{\sqrt{c+1}}\geq\frac{2\sqrt{3}}{\sqrt{a+b+c+3abc+4}}$$

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By Hölder's inequality, we have

$$\frac{a}{\sqrt{a+1}}+\frac{b}{\sqrt{b+1}}+\frac{c}{\sqrt{c+1}}\geq\sqrt{\frac{(a+b+c)^3}{a(a+1)+b(b+1)+c(c+1)}}.$$

So it suffices to prove that

$$(a+b+c)^3(a+b+c+3abc+4)\geq 12\left(a^2+b^2+c^2+a+b+c\right),$$

Let $p:=a+b+c,\ q:=ab+bc+ca,\ r:=abc$. The last inequality is equivalent to

$$f(r)=p^4+4p^3-12p^2-12p+24+3p^3r\geq 0.$$

We have $p\geq\sqrt{3q}=\sqrt{3}$, and by the fourth degree Schur's inequality, we have

$$r\geq\frac{(p^2-q)(4q-p^2)}{6p}=\frac{(p^2-1)(4-p^2)}{6p}.$$

If $p\geq 2$, we have

$$f(r)\geq f(0)=p^4+4p^3-12p^2-12p+24=(p-2)\left(p^3+6p^2-12\right)\geq 0.$$

If $\sqrt{3}\leq p\leq 2$, we have $f(r)\geq p^4+4p^3-12p^2-12p+24+\frac{p^2(p^2-1)(4-p^2)}{2}$

$$=\frac{1}{2}(2-p)\left(p-\sqrt{3}\right)\left(p^4+\left(2+\sqrt{3}\right)p^3+2\sqrt{3}p^2-8p-8\sqrt{3}\right)\geq 0.$$

Equality holds iff $a=b=c=\frac{\sqrt{3}}{3}$ & $a=b=1, c=0$ and its permutation.