ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca=1$. Prove that :

$$\sqrt{a+b-2ab}+\sqrt{b+c-2bc}+\sqrt{c+a-2ca}\geq 2\sqrt{a+b+c-1}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\left(\sqrt{a+b-2ab}+\sqrt{b+c-2bc}+\sqrt{c+a-2ca}\right)^2$$
$$\geq \frac{(2a+2b+2c-2)^3}{(a+b-2ab)^2+(b+c-2bc)^2+(c+a-2ca)^2},$$

So it suffices to prove that

$$2(a+b+c-1)^2\geq (a+b-2ab)^2+(b+c-2bc)^2+(c+a-2ca)^2,$$

which is equivalent to

$$abc[2(a+b+c)-3]\geq 0,$$

which is true since $a+b+c\geq\sqrt{3(ab+bc+ca)}=\sqrt{3}>\frac{3}{2}$.

Equality holds iff $a=b=1, c=0$ and its permutation.