ROMANIAN MATHEMATICAL MAGAZINE

**Let $a,b,c\geq 0, a+b+c=3$. Prove that :**

$$\sqrt{a^2+2abc+b^2}+\sqrt{b^2+2abc+c^2}+\sqrt{c^2+2abc+a^2}\leq 6$$

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WLOG we assume that $a=max\{a,b,c\}$. We have $1\leq a\leq 3$.

By CBS inequality, we have

$$\sqrt{a^2+2abc+b^2}+\sqrt{a^2+2abc+c^2}\leq\sqrt{2(2a^2+4abc+b^2+c^2)}$$

$$=\sqrt{4a^2+(2a+1)(b+c)^2-(2a-1)(b-c)^2}\leq\sqrt{4a^2+(2a+1)(3-a)^2}$$

$$=\sqrt{2a^3-7a^2+12a+9}.$$

Also, we have

$$2\left(b^2+2abc+c^2\right)=(a+1)(b+c)^2-(a-1)(b-c)^2\leq(a+1)(3-a)^2.$$

Let $x:=\sqrt{\frac{a+1}{2}}$. Using the last results, we only need to prove that

$$\sqrt{2a^3-7a^2+12a+9}+(3-a)\sqrt{\frac{a+1}{2}}\leq 6$$

$$\overset{a=2x^2-1}{\Leftrightarrow}\sqrt{16x^6-52x^4+64x^2-12}\leq 6-4x+2x^3$$

$$\overset{squaring}{\Leftrightarrow}0\leq 4-4x-4x^2+2x^3+3x^4-x^6$$

$$\Leftrightarrow 0\leq(1-x)^2\left(2-x^2\right)\left(2+2x+x^2\right),$$

which is true since $1\leq x\leq\sqrt{2}$, and the proof is complete.

Equality holds iff $a=b=c=1$ and $a=3,b=c=0$ and its permutation.