ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$ and $a+b+c=3$. Prove that :

$$\frac{1}{ab+3c}+\frac{1}{bc+3a}+\frac{1}{ca+3b}\geq\frac{24}{5abc+27}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{ab+3c}+\frac{1}{bc+3a}+\frac{1}{ca+3b}=$$

$$=\frac{1}{ab+c(a+b+c)}+\frac{1}{bc+a(a+b+c)}+\frac{1}{ca+b(a+b+c)}=$$

$$=\frac{1}{(c+a)(c+b)}+\frac{1}{(a+b)(a+c)}+\frac{1}{(b+c)(b+a)}=\frac{2(a+b+c)}{(a+b)(b+c)(c+a)}$$

$$=\frac{24}{4(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+8abc}\overset{Schur}{\geq}\frac{24}{(a+b+c)^3+5abc}=\frac{24}{5abc+27}.$$

Equality holds iff $a=b=c=1$ and $a=b=\frac{3}{2}, c=0$ and its permutation.