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Let $a,b,c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{a}{\sqrt{a^2+8}}+\frac{b}{\sqrt{b^2+8}}+\frac{c}{\sqrt{c^2+8}}\geq 1$$

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By CBS inequality, we have

$$\frac{a}{\sqrt{a^2+8}}+\frac{b}{\sqrt{b^2+8}}+\frac{c}{\sqrt{c^2+8}}\geq \frac{(a+b+c)^2}{a\sqrt{a^2+8}+b\sqrt{b^2+8}+c\sqrt{c^2+8}}.$$

So it suffices to prove that

$$a\sqrt{a^2+8}+b\sqrt{b^2+8}+c\sqrt{c^2+8}\leq (a+b+c)^2.$$

By AM − GM inequality and the known identity $\frac{a}{a+2}+\frac{b}{a+2}+\frac{c}{a+2}=1$, we have

$$\sum_{cyc} a\sqrt{a^2+8}\leq \sum_{cyc}\frac{a}{2}\left(\frac{a^2+8}{a+2}+a+2\right)=\sum_{cyc} a\left(a+\frac{6}{a+2}\right)=a^2+b^2+c^2+6.$$

So it suffices to prove that $3\leq ab+bc+ca$, which is true by AM − GM inequality.

The proof is complete. Equality holds iff $a=b=c=1$.