ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that :

$$\frac{1}{ab+ac+4bc}+\frac{1}{bc+ba+4ca}+\frac{1}{ca+cb+4ab}\leq\frac{3(a^2+b^2+c^2+ab+bc+ca)}{4(ab+bc+ca)^2}$$

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Multiplying the both sides of the inequality by $4(ab+bc+ca)$, we get the equivalent inequality

$$\frac{a(b+c)}{ab+ac+4bc}+\frac{b(c+a)}{bc+ba+4ca}+\frac{c(a+b)}{ca+cb+4ab}\leq\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

We have

$$\blacksquare(ab+bc+ca)\sum_{cyc}\frac{a(b+c)}{ab+ac+4bc}=\sum_{cyc}\left(a(b+c)-\frac{3abc(b+c)}{ab+ac+4bc}\right)$$

$$\overset{AM-GM}{\leq} 2(ab+bc+ca)-3abc\sum_{cyc}\frac{b+c}{ab+ac+(b+c)^2}$$

$$=2(ab+bc+ca)-\frac{9abc}{a+b+c}\overset{Schur}{\leq} a^2+b^2+c^2,$$

which completes the proof.

Equality holds iff $a=b=c$ & $a=b,c=0$ and its permutation.