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Let $a,b,c\geq 0, ab+bc+ca>0$ and $a+b+c=2$. Prove that :

$$\frac{a\sqrt{bc+b+c}}{\sqrt{a}+\sqrt{bc}}+\frac{b\sqrt{ca+c+a}}{\sqrt{b}+\sqrt{ca}}+\frac{c\sqrt{ab+a+b}}{\sqrt{c}+\sqrt{ab}}\geq 2\left(ab+bc+ca-\sqrt{abc}\right)$$

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By CBS inequality, we have

$$4(bc+b+c)=(a+b+c)^2(bc+b+c)=\left[(b+c-a)^2+4a(b+c)\right](bc+b+c)$$

$$\geq\left[(b+c-a)\sqrt{bc}+2\sqrt{a}(b+c)\right]^2.$$

Then $2\sqrt{bc+b+c}\geq(b+c-a)\sqrt{bc}+2\sqrt{a}(b+c)=2\left(\sqrt{a}+\sqrt{bc}\right)(b+c)-(a+b+c)\sqrt{bc}$

$$\Rightarrow\sum_{cyc}\frac{a\sqrt{bc+b+c}}{\sqrt{a}+\sqrt{bc}}\geq\sum_{cyc}a\left(b+c-\frac{\sqrt{bc}}{\sqrt{a}+\sqrt{bc}}\right)=2\sum_{cyc}bc-\sqrt{abc}.\sum_{cyc}\frac{\sqrt{a}}{\sqrt{a}+\sqrt{bc}}.$$

So it suffices to prove that

$$\sum_{cyc}\frac{\sqrt{a}}{\sqrt{a}+\sqrt{bc}}\leq 2\ \Leftrightarrow\ \sum_{cyc}\sqrt{a}\left(\sqrt{b}+\sqrt{ca}\right)\left(\sqrt{c}+\sqrt{ab}\right)\leq 2\left(\sqrt{a}+\sqrt{bc}\right)\left(\sqrt{b}+\sqrt{ca}\right)\left(\sqrt{c}+\sqrt{ab}\right)$$

$$\Leftrightarrow 0\leq\sqrt{abc}+2abc,$$

which is true, and the proof is complete.

Equality holds iff $a=b=1, c=0$ and its permutation.