ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{a+b}{ab+2}+\frac{b+c}{bc+2}+\frac{c+a}{ca+2}\leq\frac{2}{3}(a+b+c)$$

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Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have $4=q+r\leq\frac{p^2}{3}+\frac{p^3}{27}$, then $p\geq 3$, and

$$\sum_{cyc}\frac{a+b}{ab+2}=\frac{\sum_{cyc}(a+b)(bc+2)(ca+2)}{(ab+2)(bc+2)(ca+2)}=$$

$$=\frac{2\sum_{cyc}a^2(b+c)+12abc+2abc\sum_{cyc}bc+8\sum_{cyc}a}{8+4\sum_{cyc}bc+2abc\sum_{cyc}a+(abc)^2}=$$

$$=\frac{2pq+6r+2qr+8p}{8+4q+2pr+r^2}\overset{q=4-r}{=}\frac{16p+2(7-p)r-2r^2}{24+2(p-2)r+r^2}\overset{?}{\leq}\frac{2p}{3}$$

$$\Leftrightarrow r\left[2p^2-p-21+(p+3)r\right]\geq 0.$$

If $p\geq 4$, we have $2p^2-p-21+(p+3)r\geq 2p^2-p-21=(p-4)(2p+7)+7\geq 0$.

If $3\leq p\leq 4$, by Schur's inequality, we have $p^3+9r\geq 4pq=4p(4-r)$,

then $r\geq\frac{16p-p^3}{4p+9}$, and

$$2p^2-p-21+(p+3)r\geq 2p^2-p-21+\frac{(p+3)\left(16p-p^3\right)}{4p+9}=$$

$$=\frac{(p-3)\left(63+36p+2p^2-p^3\right)}{4p+9}\geq 0$$

Equality holds iff $(a=b=c=1)$ and $(ab=4, c=0)$ and its permutation.