ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that :

$$\frac{2c^2+2ca+2cb-ab}{a^2+b^2}+\frac{2a^2+2ab+2ac-bc}{b^2+c^2}+\frac{2b^2+2bc+2ba-ca}{c^2+a^2}\geq\frac{15}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG we assume that $a+b+c=1$. Let $p:=a+b+c=1, q:=ab+bc+ca, r:=abc$.

$$\sum_{cyc}\frac{2a^2+2ab+2ac-bc}{b^2+c^2}=\sum_{cyc}\frac{2a-bc}{b^2+c^2}=\frac{\sum_{cyc}(2a-bc)\left[a^2\left(a^2+b^2+c^2\right)+b^2c^2\right]}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$=\frac{2\sum_{cyc}a^3.\sum_{cyc}a^2+2abc\sum_{cyc}bc-abc\sum_{cyc}a^2-\sum_{cyc}(bc)^3}{\sum_{cyc}a^2.\sum_{cyc}(bc)^2-(abc)^2}$$

$$=\frac{2(1-3q+3r)(1-2q)+2qr-r(1-2q)-\left(q^3-3qr+3r^2\right)}{(1-2q)(q^2-2r)-r^2}\overset{?}{\geq}\frac{15}{2}$$

$$\Leftrightarrow f(r)=4-20q+9q^2+28q^3+(40-70q)r+9r^2\geq 0.$$

Since $3q\leq p^2=1$, then $40-70q>0$, and

If $q\leq\frac{1}{4}$, we have : $f(r)\geq f(0)=(1-4q)\left(4-4q-7q^2\right)\geq 0$.

If $\frac{1}{4}\leq q\leq\frac{1}{3}$, by the fourth degree Schur's inequality, we have

$$f(r)\geq f\left(\frac{\left(p^2-q\right)\left(4q-p^2\right)}{6p}\right)=\frac{(4q-1)(1-3q)\left(29-67q-4q^2\right)}{12}\geq 0.$$

The proof is complete. Equality holds iff $a=b=c$ & $a=b, c=0$ and its permutation.