ROMANIAN MATHEMATICAL MAGAZINE

Given real numbers a, b, c satisfying $a+b+c=3$. Prove that :

$$\sqrt{a^2+b^2+3}+\sqrt{b^2+c^2+3}+\sqrt{c^2+a^2+3}\geq\sqrt{5(a^2+b^2+c^2+6)}$$

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We assume that $a=max\{a,b,c\}$.

We have $a\geq 1$ and $b+c\leq 2$.

Let $A:=b^2+c^2+3, B:=c^2+a^2+3, C:=a^2+b^2+3$.

After squaring, the desired inequality is equivalent to

$$2\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+2\sqrt{BC}\geq 3\left(a^2+b^2+c^2+7\right)$$

$$2\sqrt{A\left(B+C+2\sqrt{BC}\right)}+2\sqrt{BC}\geq 3\left(a^2+b^2+c^2+7\right).$$

Now, we will prove that

$$\sqrt{BC}=\sqrt{(a^2+b^2+3)(a^2+c^2+3)}\geq a^2+3+\frac{b^2+c^2+bc}{3}.$$

After squaring, the inequality becomes

$$\left(3a^2+9-b^2-c^2-4bc\right)(b-c)^2\geq 0.$$

which is true since

$$3a^2+9-b^2-c^2-4bc\overset{a=3-b-c}{=}36-18(b+c)+2\left(b^2+c^2+bc\right)\geq$$

$$\geq 36-18(b+c)+\frac{3(b+c)^2}{2}=\frac{3}{2}(2-b-c)(10-b-c)+6\geq 0.$$

Therefore, it suffices to prove that

$$2\sqrt{A\left(B+C+2a^2+6+\frac{2(b^2+c^2+bc)}{3}\right)}\geq a^2+\frac{7\left(b^2+c^2\right)-2bc}{3}+15$$

$$\Leftrightarrow 12\left(b^2+c^2+3\right)\left(12a^2+36+5b^2+5c^2+2bc\right)\geq\left(3a^2+45+7b^2+7c^2-2bc\right)^2$$

Since $b^2+c^2=(b+c)^2-2bc=a^2-6a+9-2bc$, then the last inequality is equivalent to

$$12\left(a^2-6a+12-2bc\right)\left(17a^2-30a+81-8bc\right)\geq\left(10a^2-42a+108-16bc\right)^2$$

$$\Leftrightarrow f(bc)=104a^4-744a^3+1656a^2-1080a-\left(184a^2+48a-360\right)bc-64(bc)^2\geq 0.$$

f is concave and since

$$bc\leq\frac{(b+c)^2}{4}=\frac{(3-a)^2}{4}$$

$$bc=(a-b)(a-c)+a(b+c-a)\geq a(3-2a),$$

So it suffices to prove that $f\left(\frac{(3-a)^2}{4}\right)\geq 0$ and $f\left(a(3-2a)\right)\geq 0$. We have

$$f\left(\frac{(3-a)^2}{4}\right) = 54a^4 - 432a^3 + 1188a^2 - 1296a + 486 = 54(a-1)^2(3-a)^2 \geq 0.$$

$$f\big(a(3-2a)\big) = 216a^4 - 432a^3 + 216a^2 = 216a^2(a-1)^2 \geq 0.$$

Equality holds for $a = b = c = 1$ & $a = 3, b = c = 0$ and its permutation.