ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=3$. Prove that :

$$\frac{1}{a^2b+2}+\frac{1}{b^2c+2}+\frac{1}{c^2a+2}\geq\frac{7-abc}{6}$$

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Let $p:=a+b+c=3,\ \ q:=ab+bc+ca,\ \ r:=abc\leq\frac{p^3}{27}=1.$

The desired inequality is equivalent to

$$16+8r-7r^3+r^4\geq 4(1-r)\left(a^2b+b^2c+c^2a\right)+2r(4-r)\left(ab^2+bc^2+ca^2\right).\quad(1)$$

WLOG, we assume that $b=mid(a,b,c)$. By AM − GM inequality, we have

$$ab^2+bc^2+ca^2=b(c+a)^2-abc+a(b-c)(b-a)$$
$$\leq\frac{1}{2}\left(\frac{2b+(c+a)+(c+a)}{3}\right)^3-abc+0$$

then $ab^2+bc^2+ca^2\leq 4-r$. Similarly, we get $a^2b+b^2c+c^2a\leq 4-r$.

▪If $2-6r+r^2\geq 0$, we have

$$RHS_{(1)}=4(1-r)\left(a^2b+b^2c+c^2a\right)+2r(4-r)\left[pq-3r-\left(a^2b+b^2c+c^2a\right)\right]$$
$$=2\left(2-6r+r^2\right)\left(a^2b+b^2c+c^2a\right)+6r(4-r)q-6r(4-r)$$
$$\overset{Schur}{\leq}\ 2\left(2-6r+r^2\right)(4-r)+6r(4-r).\frac{p^3+9r}{4p}-6r(4-r)$$
$$\overset{p=3}{=}\frac{32-36r+61r^2-13r^3}{2}=16+8r-7r^3+r^4-\frac{r(1-r)\left(60-r-2r^2\right)}{2}$$
$$\overset{r\leq 1}{\leq}\ 16+8r-7r^3+r^4=LHS_{(1)}.$$ If $2-6r+r^2\leq 0$, we have

$$RHS_{(1)}=4(1-r)\left[pq-3r-\left(ab^2+bc^2+ca^2\right)\right]+2r(4-r)\left(ab^2+bc^2+ca^2\right)$$
$$=(1-r).4pq-2\left(2-6r+r^2\right)\left(a^2b+b^2c+c^2a\right)-12r(1-r)$$
$$\overset{Schur}{\leq}\ (1-r)\left(p^3+9r\right)-2\left(2-6r+r^2\right)(4-r)-12r(1-r)$$
$$\overset{p=3}{=}\ 11+22r-17r^2+2r^3=16+8r-7r^3+r^4-(1-r)\left[(1-r)\left(5-4r+4r^2\right)+3r^3\right]$$
$$\overset{r\leq 1}{\leq}\ 16+8r-7r^3+r^4=LHS_{(1)}.$$

Equality holds iff $(a=b=c=1)$ and $(a=2,b=1,c=0)$ and its cyclic permutations.