ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{a+bc}{b+c}+\frac{b+ca}{c+a}+\frac{c+ab}{a+b}\geq 3$$

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The desired inequality is equivalent to

$$(a+b+c+ab+bc+ca)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 6+a+b+c.$$

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have $4=q+r\leq \frac{p^2}{3}+\frac{p^3}{27}$, then $p\geq 3$.

By CBS inequality, we have

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq \frac{\left[\sum_{cyc}(a+2)\right]^2}{\sum_{cyc}(b+c)(a+2)^2}=\frac{(p+6)^2}{(pq-3r)+8q+8p}\overset{?}{\geq}\frac{p+6}{p+q}$$

$$\Leftrightarrow p^2-2p+12\geq 5q.$$

▪**If $p\geq 4$: We have $5q\leq 20=p^2-2p+12-(p-4)(p+2)\leq p^2-2p+12$.**

▪**If $3\leq p\leq 4$: We have $4pq\overset{Schur}{\leq} p^3+9r=p^3+9(4-q)$, then**

$$5q\leq \frac{5(p^3+36)}{4p+9}=p^2-2p+12-\frac{(p-3)(4-p)(p+6)}{4p+9}\leq p^2-2p+12.$$

Equality holds if $a=b=c=1$ & $a=b=2, c=0$ and permutations.