ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that

$$\sqrt{a^2+14ab+b^2}+\sqrt{b^2+14bc+c^2}+\sqrt{c^2+14ca+a^2}\geq 12\cdot\frac{ab+bc+ca}{a+b+c}$$

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Firstly, we have

$$\sqrt{b^2+14bc+c^2}-(b+c)=\frac{12bc}{\sqrt{4(b+c)^2-3(b-c)^2}+(b+c)}\geq\frac{12bc}{2(b+c)+(b+c)}=\frac{4bc}{b+c},$$

$$\Rightarrow \sqrt{b^2+14bc+c^2}\geq b+c+\frac{4bc}{b+c} \text{ (and analogs)},$$

with equality for $b=c>0$ or $bc=0$.

Using this result, it suffices to prove that

$$a+b+c+\frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ca}{c+a}\geq\frac{6(ab+bc+ca)}{a+b+c}.$$

Multiplying the last inequality by $a+b+c$, we get the equivalent form

$$a^2+b^2+c^2+2abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 2(ab+bc+ca). \quad (1)$$

By CBS and Schur inequalities, we have

$$LHS_{(1)}\geq a^2+b^2+c^2+2abc.\frac{9}{2(a+b+c)}\geq 2(ab+bc+ca).$$

So the proof is complete. Equality holds iff $a=b=c$ and

$$a=b>0, c=0 \text{ and its permutations}.$$