ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, ab+bc+ca>0$. Prove that :

$$\frac{7a^2+6ab+7b^2}{a^2+b^2}+\frac{7b^2+6bc+7c^2}{b^2+c^2}+\frac{7c^2+6ca+7a^2}{c^2+a^2}\geq\frac{90(ab+bc+ca)}{(a+b+c)^2}$$

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The desired inequality can be rewritten as follows

$$\frac{(a+b)^2}{a^2+b^2}+\frac{(b+c)^2}{b^2+c^2}+\frac{(c+a)^2}{c^2+a^2}+4\geq\frac{30(ab+bc+ca)}{(a+b+c)^2}.$$

WLOG, we assume that $a+b+c=1$.

$$\text{Let } p:=a+b+c, q:=ab+bc+ca\leq\frac{p^2}{3}=\frac{1}{3}, r:=abc.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{(b+c)^2}{b^2+c^2}.\sum_{cyc}(b^2+c^2)(5a+4)^2\geq\left(\sum_{cyc}(b+c)(5a+4)\right)^2=4(4+5q)^2,$$

So it suffices to prove that

$$\frac{4(4+5q)^2}{50\sum_{cyc}(bc)^2+40\sum_{cyc}a(b^2+c^2)+32\sum_{cyc}a^2}+4\geq 30q$$

$$\Leftrightarrow\frac{(4+5q)^2}{25(q^2-2pr)+20(pq-3r)+16(p^2-2q)}\geq 15q-2,$$

The last inequality is true for $15q-2\leq 0$. Otherwise, it is equivalent to

$$48-224q+255q^2-375q^3+110(15q-2)r\geq 0.$$

From the known identity

$$0\leq(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\geq\frac{-2p^3+9pq-2\sqrt{(p^2-3q)^3}}{27}=\frac{-2+9q-2\sqrt{(1-3q)^3}}{27}.$$

So it suffices to prove that

$$48-224q+255q^2-375q^3+\frac{110(15q-2)\left(-2+9q-2\sqrt{(1-3q)^3}\right)}{27}\geq 0$$

$$\Leftrightarrow(1-3q)(1736-6120q+3375q^2)\geq 220(15q-2)\sqrt{(1-3q)^3}$$

$$\overset{Squaring}{\Leftrightarrow}\quad 27(1-3q)^2(8-25q)^2(1632-80q+675q^2)\geq 0,$$

which is true and the proof is complete. Equality holds iff $\left(q=\frac{1}{3}\Leftrightarrow a=b=c\right)$ **and**

$$\left(q=\frac{8}{25}\Leftrightarrow a=b=2c\right)$$ **and its permutations.**