ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=3$. Prove that :

$$\sqrt{6a+6b+(a-b)^2}+\sqrt{6b+6c+(b-c)^2}+\sqrt{6c+6a+(c-a)^2}\geq 6\sqrt{3}$$

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Let $x:=b+c, y:=c+a, z:=a+b$. We have $x+y+z=6$.

The desired inequality is equivalent to

$$\sqrt{6x+(y-z)^2}+\sqrt{6y+(z-x)^2}+\sqrt{6z+(x-y)^2}\geq 6\sqrt{3}$$

$$\overset{squaring}{\Leftrightarrow}\sum_{cyc}\sqrt{[6y+(z-x)^2][6z+(x-y)^2]}\geq 3(xy+yz+zx).$$

By CBS inequality, we have

$$\sum_{cyc}\sqrt{[6y+(z-x)^2][6z+(x-y)^2]}\geq\sum_{cyc}\left[6\sqrt{yz}+(x-y)(x-z)\right]\overset{?}{\geq}3(xy+yz+zx)$$

$$\Leftrightarrow (x+y+z)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)+x^2+y^2+z^2\geq 4(xy+yz+zx)$$

which is true by summing the two following true inequalities.

$$(x+y)\sqrt{xy}+(y+z)\sqrt{yz}+(z+x)\sqrt{zx}\geq 2xy+2yz+2zx.$$

$$x^2+y^2+z^2+\sqrt{xyz}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\geq x^2+y^2+z^2+\frac{9xyz}{x+y+z}\geq 2(xy+yz+zx)$$

The proof is complete. Equality holds iff

$a=b=c=1$ & $a=b=0, c=3$ and its permutation.