ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c+abc=4$. Prove that :

$$\frac{ab}{4a+4b+c}+\frac{bc}{4b+4c+a}+\frac{ca}{4c+4a+b}\leq\frac{1}{a+b+c}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$.

$$\text{We have } 4=p+r\overset{AM-GM}{\geq} p+\frac{p^3}{27}, \text{then } 3\leq p\leq 4.$$

We have

$$\sum_{cyc}\frac{bc}{4b+4c+a}=\sum_{cyc}\frac{bc}{4p-3a}=\frac{\sum_{cyc}bc(4p-3b)(4p-3c)}{(4p-3a)(4p-3b)(4p-3c)}=$$

$$=\frac{16p^2\sum bc-12p\sum bc(b+c)+9\sum(bc)^2}{64p^3-48p^2\sum a+36p\sum bc-27abc}=\frac{16p^2q-12p(pq-3r)+9(q^2-2pr)}{64p^3-48p^3+36pq-27r}=$$

$$\overset{r=4-p}{=}\frac{4p^2q+9q^2-18p^2+72p}{16p^3+36pq+27p-108}\overset{?}{\leq}\frac{1}{p}\Leftrightarrow$$

$$4p(p^2-9)q+9pq^2\leq 34p^3-72p^2+27p-108\ (1)$$

By Schur's inequality, we have

$$q\leq\frac{p^3+9r}{4p}=\frac{p^3+9(4-p)}{4p}$$

$$LHS_{(1)}\leq(p^2-9)(p^3-9p+36)+\frac{9(p^3-9p+36)^2}{16p}=$$

$$=34p^3-72p^2+27p-108-\frac{(p-3)(4-p)(25p^4+175p^3-69p^2-207p+972)}{16p}\leq RHS_{(1)},$$

the last inequality is true because $3\leq p\leq 4$. So the proof is complete.

Equality holds iff $(a=b=c=1)$ and $(a=b=2,c=0)$ and its permutations.