ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c\geq 0, a+b+c=2$. Prove that :

$$\frac{bc}{\sqrt{5a+4}}+\frac{ca}{\sqrt{5b+4}}+\frac{ab}{\sqrt{5c+4}}\leq\frac{1}{2}$$

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We will first prove the following lemma, that

$$\frac{1}{\sqrt{5a+4}}\leq\frac{3a^2-8a+16}{32}.\ (1)$$

$$(1)\Leftrightarrow 32\leq(3a^2-8a+16)\sqrt{5a+4}\overset{squaring}{\Leftrightarrow}$$

$$0\leq a(45a^4-204a^3+608a^2-640a+256)$$

$$\Leftrightarrow 0\leq a[(5a^2-16a+13)(3a-2)^2+279a^2-420a+204],$$

which is true since $5a^2-16a+13, 279a^2-420a+204\geq 0$.

Using this lemma, we have

$$\sum_{cyc}\frac{bc}{\sqrt{5a+4}}\leq\sum_{cyc}\frac{3a^2bc-8abc+16bc}{32}=\frac{8(ab+bc+ca)-9abc}{16}=$$

$$=\frac{4(a+b+c)(ab+bc+ca)-9abc}{16}\overset{Schur}{\geq}\frac{(a+b+c)^3}{16}=\frac{1}{2}.$$

The proof is complete. Equality holds iff $a=b=1, c=0$ and its permutation.