ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c>0$. Prove that :

$$\left(\sum_{cyc}\sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc}\sqrt{\frac{a^2+ab+b^2}{3}}\right)+\frac{9(a^3+b^3+c^3)}{(a+b+c)(ab+bc+ca)}\geq 12$$

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By AM − GM inequality, we have

$$(a+b)\sqrt[3]{\frac{a^3+b^3}{2}}=2\sqrt[3]{\left(\frac{(a+b)^2}{4}\right)^2.(a^2-ab+b^2)}\leq\frac{2}{3}\left(2.\frac{(a+b)^2}{4}+(a^2-ab+b^2)\right)$$
$$=a^2+b^2$$

$$\Rightarrow\sqrt[3]{\frac{2}{a^3+b^3}}\geq\frac{a+b}{a^2+b^2}.\text{ Also, we have }\sqrt{\frac{a^2+ab+b^2}{3}}=\sqrt{\frac{(a+b)^2}{4}+\frac{(a-b)^2}{12}}\geq\frac{a+b}{2}.$$

If $a\geq b\geq c$, we have $\frac{a+b}{2}\geq\frac{c+a}{2}\geq\frac{b+c}{2}$ and $\sqrt[3]{\frac{2}{a^3+b^3}}\leq\sqrt[3]{\frac{2}{c^3+a^3}}\leq\sqrt[3]{\frac{2}{b^3+c^3}}$, then by

Chebyshev and CBS inequalities, we have

$$\left(\sum_{cyc}\sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc}\sqrt{\frac{a^2+ab+b^2}{3}}\right)\geq\left(\sum_{cyc}\sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc}\frac{a+b}{2}\right)\geq 3\sum_{cyc}\frac{a+b}{2}.\sqrt[3]{\frac{2}{a^3+b^3}}$$
$$\geq\frac{3}{2}\sum_{cyc}\frac{(a+b)^2}{a^2+b^2}\geq\frac{3(a+b+c)^2}{a^2+b^2+c^2}=3+\frac{6(ab+bc+ca)}{a^2+b^2+c^2}.\quad(1)$$

Now, by Schur's inequality, we have

$$\frac{3(a^3+b^3+c^3)}{(a+b+c)(ab+bc+ca)}\geq\frac{2(a^3+b^3+c^3)+a^2(b+c)+b^2(c+a)+c^2(a+b)-3abc}{(a+b+c)(ab+bc+ca)}$$
$$=\frac{(a+b+c)(2a^2+2b^2+2c^2-ab-bc-ca)}{(a+b+c)(ab+bc+ca)}=\frac{2(a^2+b^2+c^2)}{ab+bc+ca}-1.\quad(2)$$

From (1) and (2), it suffices to prove that

$$\frac{6(ab+bc+ca)}{a^2+b^2+c^2}+\frac{6(a^2+b^2+c^2)}{ab+bc+ca}\geq 12,$$

which is true by AM − GM inequlity.

So the proof is complete. Equality holds iff $a=b=c$.