ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$, $x + y = 2$ then:

$$\frac{x}{\sqrt{y^3+1}} + \frac{y}{\sqrt{x^3+1}} \geq \sqrt{2}$$

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$$(x^2y^3 + x^2) + (x^3y^2 + y^2) = x^2y^2(x+y) + (x^2+y^2) =$$

$$= x^2y^2(x+y) + (x+y)^2 - 2xy \overset{x+y=2}{=} 2x^2y^2 + (x+y)^2 - 2xy =$$

$$= 2xy\,(xy) + (x+y)^2 - 2xy \overset{Am-Gm}{\leq} 2xy\frac{(x+y)^2}{4} + (x+y)^2 - 2xy \overset{x+y=2}{=}$$

$$= 2xy.\frac{2^2}{4} + (x+y)^2 - 2xy = (x+y)^2 = 2^2 = 4\ (1)$$

$$\frac{x}{\sqrt{y^3+1}} + \frac{y}{\sqrt{x^3+1}} = \frac{x^2}{x\sqrt{y^3+1}} + \frac{y^2}{y\sqrt{x^3+1}} \overset{Bergstrom}{\geq} \frac{(x+y)^2}{\sqrt{x^2y^3+x^2} + \sqrt{y^2x^3+y^2}} \overset{CBS}{\geq}$$

$$\geq \frac{(x+y)^2}{\sqrt{2\left((x^2y^3+x^2) + (x^3y^2+y^2)\right)}} \overset{(1)}{\geq} \frac{(x+y)^2}{\sqrt{2\times 4}} \overset{x+y=2}{=} \frac{4}{2\sqrt{2}} = \sqrt{2}$$

Equality holds for $x = y = 1$.