ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ and $xyz=1$, then prove that :

$$2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq 3(x+y+z)+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

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$$2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq 3(x+y+z)+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

$$\Leftrightarrow 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6\geq 3\sum_{cyc}x+\sum_{cyc}xy\ (\because xyz=1)$$

$$\Leftrightarrow \left(\sum_{cyc}x\right)\left(2\sum_{cyc}xy-3\right)\overset{(*)}{\geq}\sum_{cyc}xy+6 \text{ and } \because \sum_{cyc}xy\overset{AM-GM}{\geq}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{=}3$$

$$\therefore 2\sum_{cyc}xy-3\geq 3>0 \text{ and } \because \sum_{cyc}x\geq\sqrt{3\sum_{cyc}xy}$$

$$\therefore \text{LHS of } (*)\geq\sqrt{3m}.(2m-3)\overset{?}{\geq}m+6\left(m=\sum_{cyc}xy\right)$$

$$\Leftrightarrow 3m(2m-3)^2-(m+6)^2\overset{?}{\geq}0\Leftrightarrow(m-3)(12m^2-m+12)\overset{?}{\geq}0\rightarrow \text{true}$$

$$\because m\geq 3\Rightarrow (*) \text{ is true } \therefore 2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq 3(x+y+z)+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

$$\forall x,y,z>0\,|\,xyz=1,'' ='' \text{ iff } x=y=z=1 \text{ (QED)}$$