ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^3+b^3+c^3=ab+bc+ca$, then prove that :

$$a^2+b^2+c^2\le 3$$

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$$a^2+b^2+c^2\le 3\Leftrightarrow \sum_{cyc} a^2\le 3\left(\frac{\sum_{cyc} a^3}{\sum_{cyc} ab}\right)^2\left(\because 1=\frac{\sum_{cyc} a^3}{\sum_{cyc} ab}\right)$$

$$\Leftrightarrow 3\left(\sum_{cyc} a^3\right)^2\ge\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2\to(1)$$

$$\text{Now, } 3\left(\sum_{cyc} a^3\right)^2\overset{\text{Chebyshev}}{\ge}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^3\right)$$

$$=\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)\right)\left(\sum_{cyc} a^2\right)\overset{\text{Reverse CBS}}{\ge}\left(\sum_{cyc} a^2\right)^2.\left(\sum_{cyc} a^2\right)$$

$$\ge\left(\sum_{cyc} ab\right)^2.\left(\sum_{cyc} a^2\right)\Rightarrow(1)\text{ is true }\therefore a^2+b^2+c^2\le 3$$

$$\forall\, a,b,c>0\,\big|\,a^3+b^3+c^3=ab+bc+ca,''='' \text{ iff } a=b=c=1\ (\text{QED})$$