# ROMANIAN MATHEMATICAL MAGAZINE

**If $x,y,z>0$, then prove that :**

$$\frac{x(y+z)}{y^2+yz+z^2}+\frac{y(z+x)}{z^2+zx+x^2}+\frac{z(x+y)}{x^2+xy+y^2}\geq 2$$

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Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0$,
$b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$
$\Rightarrow a,b,c$ formsides of $a$ triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $\sum_{cyc} x \overset{(\bullet)}{=} s, xyz \overset{(\bullet\bullet)}{=} r^2s, \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr+r^2$,

$$\sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2-8Rr-2r^2, \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R+r)^2-2s^2),$$

$$\sum_{cyc} x^3 \overset{(\bullet\bullet\bullet\bullet\bullet\bullet)}{=} s(s^2-12Rr) \text{ and } \frac{x(y+z)}{y^2+yz+z^2}+\frac{y(z+x)}{z^2+zx+x^2}+\frac{z(x+y)}{x^2+xy+y^2}$$

$$=\frac{\sum_{cyc}\left(x(y+z)(z^2+zx+x^2)(x^2+xy+y^2)\right)}{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

$$=\frac{\left(\begin{array}{c}(\sum_{cyc} xy)(\sum_{cyc} x^2)^2+(\sum_{cyc} x^2-2\sum_{cyc} xy)(\sum_{cyc} x^2y^2)+xyz\sum_{cyc} x^3+2(\sum_{cyc} xy)^3- \\ 2xyz(\sum_{cyc} x)(\sum_{cyc} xy)-3x^2y^2z^2\end{array}\right)}{(\sum_{cyc} x^2)(\sum_{cyc} x^2y^2)+(\sum_{cyc} xy)^3-xyz(\sum_{cyc} x)(\sum_{cyc} xy)-3x^2y^2z^2+xyz\sum_{cyc} x^3}$$

$$\geq 2 \Leftrightarrow (2R+r)s^4-r(40R^2+14Rr+r^2)s^2+2r^2(4R+r)^3 \overset{(*)}{\geq} 0$$

(on plugging in $(\bullet),(\bullet\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet\bullet)$ and post simplification)
and $\because P=(2R+r)(s^2-16Rr+5r^2)^2+$
$r(24R^2-2Rr-11r^2)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$,
it suffices to prove : LHS of $(*) \overset{?}{\geq} P \Leftrightarrow 8r^3(R-2r)^2 \overset{?}{\geq} 0 \rightarrow$ true $\Rightarrow (*)$ is true

$$\therefore \frac{x(y+z)}{y^2+yz+z^2}+\frac{y(z+x)}{z^2+zx+x^2}+\frac{z(x+y)}{x^2+xy+y^2}\geq 2 \ \forall x,y,z>0,$$

$''='' \text{ iff } x=y=z$ (QED)