ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\geq\frac{10}{3}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a=s\to(1), abc=r^2s\to(2)$,

$$\sum_{cyc} a^2=s^2-8Rr-2r^2\to(3), \sum_{cyc} a^2b^2=r^2((4R+r)^2-2s^2)\to(4)$$

$$\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\geq\frac{10}{3} \overset{a+b+c=3}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)+\left(\sum_{cyc} a\right)^2 .a^2b^2c^2\geq 30a^2b^2c^2\left(\sum_{cyc} a^2\right)$$

$$\overset{via\ (1),(2),(3)\ and\ (4)}{\Leftrightarrow} (s^2-8Rr-2r^2)((4R+r)^2-2s^2)+s^2r^2-$$

$$30r^2(s^2-8Rr-2r^2)\overset{(*)}{\boxed{\geq}}0 \text{ and } \because P=-2s^2(s^2-4R^2-4Rr-3r^2)+$$

$$2(4R^2+8Rr-15r^2)(s^2-16Rr+5r^2)\overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (*),$$

it suffices to prove : LHS of $(*)\overset{?}{\geq} P \Leftrightarrow 8r^2(15R-13r)(R-2r)\overset{?}{\geq}0$

$\to$ true via Euler $\Rightarrow (*)$ is true $\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\geq\frac{10}{3}$

$\forall a,b,c>0 \mid a+b+c=3,''='' $ iff $a=b=c=1$ (QED)