ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$(a^2+b^2+c^2)^3 \geq 9(a^3+b^3+c^3)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $abc=r^2s \rightarrow (1), \sum_{cyc} a^2=s^2-8Rr-2r^2 \rightarrow (2)$ and

$$\sum_{cyc} a^3=s^3-12Rrs \rightarrow (3) \therefore (a^2+b^2+c^2)^3 \geq 9(a^3+b^3+c^3) \overset{abc=1}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a^2\right)^3 \geq 9abc\left(\sum_{cyc} a^3\right) \overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}$$

$$(s^2-8Rr-2r^2)^3-9r^2s(s^3-12Rrs) \overset{(*)}{\geq} 0 \text{ and } \because P=$$

$$(s^2-16Rr+5r^2)^3+(24Rr-30r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*) \overset{?}{\geq} P$

$$\Leftrightarrow (192R^2-516Rr+237r^2)s^2 \overset{?}{\underset{(**)}{\geq}} r(2560R^3-7296R^2r+4296Rr^2-617r^3)$$

Case 1 $192R^2-516Rr+237r^2 \geq 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(192R^2-516Rr+237r^2)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 128t^3-480t^2+519t-142 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2-224t+71) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true}$$

Case 2 $192R^2-516Rr+237r^2<0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(192R^2-516Rr+237r^2)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 192t^4-964t^3+1689t^2-1224t+332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(192t^2-196t+137)+108\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true $\forall \Delta ABC$

$\therefore (a^2+b^2+c^2)^3 \geq 9(a^3+b^3+c^3) \; \forall a,b,c>0 \mid abc=1$,

$''='' $ iff $a=b=c=1$ (QED)