ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0,\ xy = 1$ then:

$$\frac{x^4}{y^3+1} + \frac{y^4}{x^3+1} \geq 1$$

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$$(x^7+y^7)+(x^4+y^4) = (x^{4+3}+y^{4+3})+(x^4+y^4) \overset{Chebyshev\ \&\ AM-GM}{\geq}$$

$$\geq \frac{1}{2}(x^3+y^3)(x^4+y^4) + 2(x^4y^4)^{\frac{1}{2}} \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{2}(x^3+y^3)2(x^4y^4)^{\frac{1}{2}} + 2(x^4y^4)^{\frac{1}{2}} \overset{xy=1}{=} (x^3+y^3)+2\ (1)$$

$$\frac{x^4}{y^3+1} + \frac{y^4}{x^3+1} = \frac{(x^7+y^7)+(x^4+y^4)}{x^3y^3+(x^3+y^3)+1} \overset{xy=1}{=}$$

$$= \frac{(x^7+y^7)+(x^4+y^4)}{2+(x^3+y^3)} \overset{(1)}{\geq} \frac{(x^3+y^3)+2}{(x^3+y^3)+2} = 1$$

Equality holds for $x = y = 1$.