ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

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Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r\ (say)$; and then : $\sum_{cyc} a=s\rightarrow(1), \sum_{cyc} ab=4Rr+r^2\rightarrow(2)$ and

$$\sum_{cyc} a^2=s^2-8Rr-2r^2\rightarrow(3)\ \&\ \text{now},\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\Leftrightarrow\frac{7}{2}+\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}\geq\left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{b+c}\right)\overset{via\ (1),(2)\ and\ (3)}{\Leftrightarrow}\frac{7}{2}+\frac{s^2-8Rr-2r^2}{4Rr+r^2}$$

$$\geq s.\frac{s^2+4Rr+r^2}{4Rrs}\Leftrightarrow s^2-(8R^2-2Rr-r^2)\leq 0\Leftrightarrow$$

$$\left(s^2-(4R^2+4Rr+3r^2)\right)-2(R-2r)(2R+r)\leq 0\rightarrow \text{true via Gerretsen \& Euler}$$

$$\therefore\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$