ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0,\ xy\sqrt{xy}+yz\sqrt{yz}+zx\sqrt{zx}=1$ then:

$$\frac{x^6}{x^3+y^3}+\frac{y^6}{y^3+z^3}+\frac{z^6}{z^3+x^3}\geq\frac{1}{2}$$

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Solution by Tapas Das-India

$$1=\ xy\sqrt{xy}+yz\sqrt{yz}+zx\sqrt{zx}=$$

$$\sum xy\sqrt{xy}=\sum\sqrt{x^3y^3}\overset{CBS}{\leq}\sqrt{3(x^3y^3+y^3z^3+z^3x^3)}\leq$$

$$\leq\sqrt{\frac{3(x^3+y^3+z^3)^2}{3}}=x^3+y^3+z^3\ (1)$$

$$\frac{x^6}{x^3+y^3}+\frac{y^6}{y^3+z^3}+\frac{z^6}{z^3+x^3}=\sum\frac{x^6}{x^3+y^3}=$$

$$=\sum\left(x^3-\frac{x^3y^3}{x^3+y^3}\right)=\sum x^3-\sum\frac{x^3y^3}{x^3+y^3}\geq$$

$$\overset{AM-GM}{\geq}\sum x^3-\sum\frac{x^3y^3}{2\sqrt{x^3y^3}}=\sum x^3-\frac{1}{2}\sum\sqrt{x^3y^3}\overset{(1)}{\leqq}1-\frac{1}{2}=\frac{1}{2}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{\sqrt[3]{3}}$$