ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca=1$, then prove that:

$$\frac{(a^2+1)^2}{bc(b+c)}+\frac{(b^2+1)^2}{ca(c+a)}+\frac{(c^2+1)^2}{ab(a+b)}\geq 8\sqrt{3}$$

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Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a=s\Rightarrow a=s-x, b=s-y, c=s-z\rightarrow(1)$,

$$abc=r^2s\rightarrow(2),\ \sum_{cyc} ab=4Rr+r^2\rightarrow(3)$$

$$\text{Now, }\frac{(a^2+1)^2}{bc(b+c)}+\frac{(b^2+1)^2}{ca(c+a)}+\frac{(c^2+1)^2}{ab(a+b)}\overset{ab+bc+ca=1}{=}\sum_{cyc}\frac{a(a^2+ab+bc+ca)^2}{abc(b+c)}$$

$$=\sum_{cyc}\frac{a(a+b)^3(c+a)^3}{abc(a+b)(b+c)(c+a)}\overset{\text{via (1) and (2) and } ab+bc+ca=1}{=}\sum_{cyc}\frac{y^3z^3(s-x)}{r^2s.4Rrs}.\frac{1}{\sqrt{\sum_{cyc} ab}}$$

$$\overset{\text{via (3)}}{=}\frac{1}{4Rr^3s^2.\sqrt{4Rr+r^2}}.\left(s\sum_{cyc}x^3y^3-4Rrs\sum_{cyc}x^2y^2\right)$$

$$\overset{\text{Gerretsen + Euler}}{\geq}\frac{1}{4Rr^3s.\sqrt{\frac{s^2}{3}}}.\begin{pmatrix}(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\\-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)\end{pmatrix}\overset{?}{\geq}8\sqrt{3}$$

$$\Leftrightarrow (s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$-32Rr^3s^2\overset{?}{\geq}0 \text{ and } \because P=s^4(s^2-16Rr+5r^2)-2r^2s^2(s^2-4R^2-4Rr-3r^2)$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{ in order to prove } (*), \text{ it suffices to prove : LHS of } (*)\overset{?}{\geq}P$$

$$\Leftrightarrow (24R^2-48Rr-3r^2)s^2+r^2(4R+r)^2\overset{?}{\geq}0 \text{ and } (**) \text{ is trivially true if :}$$

$$24R^2-48Rr-3r^2\geq 0 \text{ and so, we now consider the case when :}$$

$$24R^2-48Rr-3r^2<0; \text{ and then : LHS of } (**)\overset{\text{Gerretsen}}{\geq}$$

$$(24R^2-48Rr-3r^2)(4R^2+4Rr+3r^2)+r^2(4R+r)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 24t^4-24t^3-29t^2-37t-2\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(24t^3+24t^2+19t+1) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{(a^2+1)^2}{bc(b+c)} + \frac{(b^2+1)^2}{ca(c+a)} + \frac{(c^2+1)^2}{ab(a+b)} \geq 8\sqrt{3}\ \forall a,b,c>0 \,\big|\, ab+bc+ca=1,$$

$$'' ='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}} \ (\text{QED})$$