ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2$$

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$\boldsymbol{Lemma}$:

$$\frac{t}{2\sqrt{t}+t}\leq\frac{2+t}{9}\ \forall\ t>0$$

$\boldsymbol{Proof}$:

$$\frac{t}{2\sqrt{t}+t}\leq\frac{2+t}{9}\ \ or\ (2+t)\left(2\sqrt{t}+t\right)\geq 9t\ \ or\ (2+m^2)(2m+m^2)\overset{t=m^2}{\geq}\ 9m^2$$

$$or\ m^4+2m^3-7m^2+4m\geq 0\ or\ m(m-1)^2(m+4)\geq 0\ true$$

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}=\sum\frac{a^2(b+1)}{a+b+ab}=$$

$$=\sum\frac{a^2(b+1)}{a(b+1)+b}=\sum\left(a-\frac{ab}{a(b+1)+b}\right)=$$

$$=\sum a-\sum\frac{ab}{ab+a+b}\overset{AM-GM}{\geq}\sum a-\sum\frac{ab}{ab+2\sqrt{ab}}\overset{Lemma}{\geq}$$

$$\sum a-\sum\frac{2+ab}{9}\overset{a+b+c=3}{=}3-\frac{6+ab+bc+ca}{9}\geq$$

$$\geq 3-\frac{6+\frac{(a+b+c)^2}{3}}{9}=3-\frac{6+\frac{9}{3}}{9}=3-1=2$$

Equality holds for $a=b=c=1$.