ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

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$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\sum_{cyc}\frac{b+c}{a}+2\sum_{cyc}\frac{b}{a}\geq$$

$$\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-3abc}{abc}+\frac{2\sum_{cyc}a}{\sqrt[3]{abc}}\overset{abc=1}{=}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3+2\sum_{cyc}a\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{abc}.\left(\sum_{cyc}ab\right)-3+2\sum_{cyc}a\overset{abc=1}{=}$$

$$=2\sum_{cyc}ab+\sum_{cyc}ab-3+2\sum_{cyc}a\overset{abc=1}{=}$$

$$=\frac{2\sum_{cyc}ab}{abc}+2\sum_{cyc}a+\sum_{cyc}ab-3=$$

$$=2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\sum_{cyc}ab-3\overset{AM-GM}{\geq}2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$+3\sqrt[3]{a^2b^2c^2}-3\overset{abc=1}{=}2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\therefore\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\forall\, a,b,c>0\,|\,abc=1,''='' \text{ iff } a=b=c=1\ (\text{QED})$$