ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0,\ x+y+z=6$ then:

$$\frac{xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\le 2\sqrt{2}$$

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$$\textbf{Lemma}:\forall\ a,b,c\in R^{+}\ \textit{such that}\ a+b+c=3\ \textit{then}\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ge ab+bc+ca$$

$$\textbf{Proof}:3(a+b+c)=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$\textbf{Hence}:\sum ab=\frac{1}{2}(3a-a^2+3b-b^2+3c-c^2)$$

$$\textbf{then}\ \sqrt{a}+\sqrt{b}+\sqrt{c}-(ab+bc+ca)$$

$$=\sqrt{a}+\sqrt{b}+\sqrt{c}-\frac{1}{2}(3a-a^2+3b-b^2+3c-c^2)=\frac{1}{2}\sum\left(a^2-3a+2\sqrt{a}\right)=$$

$$=\frac{1}{2}\sum\left(\sqrt{a}\left(\sqrt{a}-1\right)^2\left(\sqrt{a}+2\right)\right)\ge 0\ \textit{true}$$

$$\textbf{Back to the main problem}\ \ x+y+z=6\ \textit{or}\ \frac{x}{2}+\frac{y}{2}+\frac{z}{2}=3$$

$$\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{2}}+\sqrt{\frac{z}{2}}\overset{Lemma}{\ge}\frac{xy}{4}+\frac{yz}{4}+\frac{zx}{4}\ \textit{or}$$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\ge\frac{xy+yz+zx}{2\sqrt{2}}\ \textit{or}\ \frac{xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\le 2\sqrt{2}$$

Equality holds for $x=y=z=2$.