ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then:

$$\sqrt{\frac{x}{z+3x}} + \sqrt{\frac{y}{x+3y}} + \sqrt{\frac{z}{y+3z}} \leq \frac{3}{2}$$

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$$\sum \frac{x}{z+3x} = \frac{1}{3}\sum\left(1 - \frac{z}{z+3x}\right) = \frac{1}{3}\left(3 - \sum \frac{z}{z+3x}\right) =$$

$$= \frac{1}{3}\left(3 - \sum \frac{z^2}{z^2+3xz}\right) \overset{CBS}{\leq} \frac{1}{3}\left(3 - \frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}\right) =$$

$$= \frac{1}{3}\left(2 + 1 - \frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}\right) =$$

$$= \frac{1}{3}\left(2 + \frac{xy+yz+zx}{x^2+y^2+z^2+3(xy+yz+zx)}\right) =$$

$$\leq \frac{1}{3}\left(2 + \frac{xy+yz+zx}{xy+yz+zx+3(xy+yz+zx)}\right) = \frac{1}{3}\left(2 + \frac{1}{4}\right) = \frac{1}{3}\cdot\frac{9}{4} \ (1)$$

$$\sqrt{\frac{x}{z+3x}} + \sqrt{\frac{y}{x+3y}} + \sqrt{\frac{z}{y+3z}} \overset{CBS}{\leq} \sqrt{3\sum \frac{x}{z+3x}} \overset{(1)}{\leq} \sqrt{\frac{3,1}{3}\cdot\frac{9}{4}} = \frac{3}{2}$$

Equality holds for $x = y = z$.