ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\ a^2+b^2+c^2=3$ then:

$$\frac{a^3+b^3}{a+2b}+\frac{b^3+c^3}{b+2c}+\frac{c^3+a^3}{c+2a}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a^3}{a+2b}=\sum\left(a^2-\frac{2a^2b}{a+2b}\right)=$$

$$=\sum a^2-2\sum\frac{a^2b}{a+b+b}\overset{AM-HM}{\geq}\sum a^2-\frac{2}{9}\sum(ab+a^2+a^2)=$$

$$=\sum a^2-\frac{2}{9}\sum ab-\frac{4}{9}\sum a^2\geq\sum a^2-\frac{2}{9}\sum a^2-\frac{4}{9}\sum a^2=\frac{1}{3}\sum a^2=\frac{1}{3}.3=1$$

$$\sum\frac{b^3}{a+2b}=\frac{1}{2}\sum\left(b^2-\frac{ab^2}{a+b+b}\right)=\frac{1}{2}\sum b^2-\frac{1}{2}\sum\frac{ab^2}{a+b+b}\overset{AM-HM}{\geq}$$

$$\geq\frac{1}{2}\sum b^2-\frac{1}{2}\cdot\frac{1}{9}\left(\sum b^2+\sum ab+\sum ab\right)\geq$$

$$\geq\frac{1}{2}\sum b^2-\frac{1}{18}\left(\sum b^2+\sum b^2+\sum b^2\right)=\frac{1}{3}\sum b^2=\frac{1}{3}.3=1$$

$$\frac{a^3+b^3}{a+2b}+\frac{b^3+c^3}{b+2c}+\frac{c^3+a^3}{c+2a}=\sum\frac{a^3+b^3}{a+2b}=\sum\frac{a^3}{a+2b}+\sum\frac{b^3}{a+2b}\geq 1+1=2$$

Equality holds for $a=b=c=1$.