ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\geq 1$$

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$$WLOG\ a\geq b\geq c\ then\ a+b\geq a+c\geq b+c\ and\ \frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$$

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\overset{Chebyshev}{\geq}\frac{1}{2}(a+b)\left(\frac{1}{b+c}+\frac{1}{c+a}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{2}(a+b)\cdot 2\sqrt{\frac{1}{(b+c)(c+a)}}=\frac{a+b}{\sqrt{(b+c)(c+a)}}$$

$$Similarly:$$

$$\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\geq\frac{b+c}{\sqrt{(b+a)(c+a)}},$$

$$\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\geq\frac{c+a}{\sqrt{(b+a)(c+b)}}$$

$$Using\ above\ result\ we\ get:$$

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\geq$$

$$\geq\frac{a+b}{\sqrt{(b+c)(c+a)}}\cdot\frac{b+c}{\sqrt{(b+a)(c+a)}}\cdot\frac{c+a}{\sqrt{(b+a)(c+b)}}=1$$

Equality holds for a=b=c=1.