ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ *and* $a^2+b^2=c^2$ then:

$$\frac{(a+b)(b+c)(c+a)}{abc}\geq 4+3\sqrt{2}$$

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$$Let\ a=c\cos\theta\,,b=c\sin\theta$$

$$Let\ t=\cos\theta+\sin\theta=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)=$$

$$=\sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right)\leq\sqrt{2}\ \left(as\ \sin\left(\frac{\pi}{4}+\theta\right)\leq 1\right)$$

$$\sin\theta\,.\cos\theta=\frac{(\cos\theta+\sin\theta)^2-1}{2}=\frac{t^2-1}{2}$$

$$Putting\ the\ value\ of\ above\ mentioned\ a,c\ we\ get:$$

$$\frac{(a+b)(b+c)(c+a)}{abc}=\frac{(\cos\theta+\sin\theta)(1+\sin\theta)(1+\cos\theta)}{\sin\theta\,.\cos\theta}$$

$$=\frac{(\cos\theta+\sin\theta)(1+\sin\theta+\cos\theta+\sin\theta\cos\theta)}{\sin\theta\,.\cos\theta}=\frac{t\left(1+t+\frac{t^2-1}{2}\right)}{\frac{t^2-1}{2}}=$$

$$=\frac{t(t+1)^2}{t^2-1}=\frac{t(t+1)}{t-1}$$

$$(putting\ the\ value\ of\ (\cos\theta+\sin\theta)\&\ \ \cos\theta\,.\sin\theta)$$

$$We\ need\ to\ show:$$

$$\frac{t(t+1)}{t-1}\geq 4+3\sqrt{2}\ or\ t^2-t\left(3+3\sqrt{2}\right)+\left(4+3\sqrt{2}\right)\geq 0$$

$$or\ \left(t-\sqrt{2}\right)\left(t-\left(3+2\sqrt{2}\right)\right)\geq 0\ true\ \left(as\ t\leq\sqrt{2}\right)$$

Equality holds for a=b =x and c=$\sqrt{2}x$ *where* $x>0$.