ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$, $a+b=1$ then:

$$\frac{1}{a^2+ab+b^2}+\frac{4a^2b^2+1}{ab}\geq\frac{19}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ m=ab\overset{AM-GM}{\leq}\frac{(a+b)^2}{4}\overset{a+b=1}{=}\frac{1}{4}\ (1)$$

$$a^2+ab+b^2=(a+b)^2-2ab+ab=(1-m)\ (2)$$

$$We\ need\ to\ show:$$

$$\frac{1}{a^2+ab+b^2}+\frac{4a^2b^2+1}{ab}\geq\frac{19}{3}\ or$$

$$\frac{1}{1-m}+\frac{4m^2+1}{m}\geq\frac{19}{3}\ or\ \frac{1+4m^2-4m^3}{m(1-m)}\geq\frac{19}{3}$$

$$12m^3-31m^2+19m-3\leq 0\ or(4m-1)\{(m-2)(3m-1)+1\}\leq 0$$

$$true\ since\ m\leq\frac{1}{4}\ then\ (m-2)<0\ and\ (3m-1)=3\left(m-\frac{1}{3}\right)<0$$

$$(4m-1)<0\ \ then(4m-1)\{(m-2\ (m-2)(3m-1)+1\}<0$$

Equality holds for $a=b=\frac{1}{2}$.