ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$(a+b+c)^3(a+b-c)(b+c-a)(c+a-b)\le 27a^2b^2c^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a=s\to(1), a=s-x, b=s-y, c=s-z\to(2)$,

$$abc=r^2s\to(3) \text{ and, } (a+b+c)^3(a+b-c)(b+c-a)(c+a-b) \overset{\text{via (1) and (2)}}{=} s^3\big(x-(s-x)\big)\big(y-(s-y)\big)\big(z-(s-z)\big)=s^3(2x-s)(2y-s)(2z-s)$$

$$=s^3\left(-s^3+2s^2\sum_{cyc}x-4s\sum_{cyc}xy+8xyz\right)$$

$$=s^3(-s^3+2s^2(2s)-4s(s^2+4Rr+r^2)+32Rrs)=-s^4(s^2-16Rr+4r^2)$$

$$\therefore RHS-LHS \overset{\text{via (3)}}{=} 27r^4s^2+s^4(s^2-16Rr+4r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow s^4-(16Rr-4r^2)s^2+27r^4\overset{?}{\ge}0 \ (*) \text{ and } \because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0,$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (8R-3r)s^2\overset{?}{\ge}r(128R^2-80Rr-r^2) \ (**)$$

$$\text{Now, } (8R-3r)s^2 \overset{\text{Rouche}}{\ge} (8R-3r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\ge} r(128R^2-80Rr-r^2)$$

$$\Leftrightarrow 2(R-2r)(8R^2-11Rr-r^2)\overset{?}{\ge}2(R-2r)(8R-3r).\sqrt{R^2-2Rr} \ (***) \text{ and } \because$$

$$R-2r \overset{\text{Euler}}{\ge} 0 \therefore \text{in order to prove } (***), \text{ it suffices to prove :}$$

$$(8R^2-11Rr-r^2)^2\overset{?}{>}(R^2-2Rr)(8R-3r)^2 \Leftrightarrow r^3(40R+r)\overset{?}{>}0\to\text{true}$$

$\therefore (***)\Rightarrow(**)\Rightarrow(*)$ is true $\therefore (a+b+c)^3(a+b-c)(b+c-a)(c+a-b)$
$\le 27a^2b^2c^2 \ \forall a,b,c>0,''='' \text{ iff } a=b=c$ (QED)