ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$(a^2+b^2+c^2)(b+c-a)(c+a-b)(a+b-c)\le abc(ab+bc+ca)$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius R and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a = s \to (1), a=s-x, b=s-y, c=s-z \to (2)$,

$$abc=r^2s\to(3),\ \sum_{cyc}ab=4Rr+r^2\to(4),\ \sum_{cyc}a^2=s^2-8Rr-2r^2\to(5)$$ and so,

$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \overset{\text{via (1),(2) and (5)}}{=} (s^2-8Rr-2r^2)\big(x-(s-x)\big)\big(y-(s-y)\big)\big(z-(s-z)\big)$$
$$=(s^2-8Rr-2r^2)(2x-s)(2y-s)(2z-s)$$
$$=(s^2-8Rr-2r^2)\left(-s^3+2s^2\sum_{cyc}x-4s\sum_{cyc}xy+8xyz\right)$$
$$=(s^2-8Rr-2r^2)(-s^3+2s^2(2s)-4s(s^2+4Rr+r^2)+32Rrs)$$
$$=-s(s^2-8Rr-2r^2)(s^2-16Rr+4r^2) \therefore RHS-LHS \overset{\text{via (3) and (4)}}{=} r^2s(4Rr+r^2)+s(s^2-8Rr-2r^2)(s^2-16Rr+4r^2) \overset{?}{\ge} 0$$
$$\Leftrightarrow s^4-(24Rr-2r^2)s^2+r^2(128R^2+4Rr-7r^2) \overset{?}{\underset{(*)}{\ge}} 0 \text{ and}$$

$\because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{\ge} (s^2-16Rr+5r^2)^2 \Leftrightarrow (2R-2r)s^2 \overset{?}{\underset{(**)}{\ge}} r(32R^2-41Rr+8r^2)$$

$$\text{Now, } (2R-2r)s^2 \overset{\text{Rouche}}{\ge} (2R-2r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right) \overset{?}{\ge} r(32R^2-41Rr+8r^2)$$
$$\Leftrightarrow (R-2r)(4R^2-8Rr+3r^2) \overset{?}{\underset{(***)}{\ge}} 2(R-2r)(2R-2r).\sqrt{R^2-2Rr} \text{ and } \because$$
$$R-2r \overset{\text{Euler}}{\ge} 0 \therefore \text{in order to prove } (***), \text{ it suffices to prove :}$$
$$(4R^2-8Rr+3r^2)^2 \overset{?}{>} 4(R^2-2Rr)(2R-2r)^2 \Leftrightarrow r^2(8R(R-2r)+9r^2) \overset{?}{>} 0$$
$\to$ true $\therefore (***) \Rightarrow (**) \Rightarrow (*)$ is true
$\therefore (a^2+b^2+c^2)(b+c-a)(c+a-b)(a+b-c)\le abc(ab+bc+ca)$
$\forall a,b,c>0$, "$=$" iff $a=b=c$ (QED)