ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=1$, then prove that :

$$\sqrt{2-(ab+bc+ca)+3abc}\geq 2\sqrt[3]{(a+b)(b+c)(c+a)}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a=s\rightarrow(1), abc=r^2s\rightarrow(2)$,

$$\sum_{cyc} ab=4Rr+r^2\rightarrow(3), \prod_{cyc}(b+c)=4Rrs\rightarrow(4)$$

$$\text{Now, } \sqrt{2-(ab+bc+ca)+3abc}\geq 2\sqrt[3]{(a+b)(b+c)(c+a)} \overset{a+b+c=1}{\Leftrightarrow}$$

$$\left(2\left(\sum_{cyc} a\right)^2-\sum_{cyc} ab+\frac{3abc}{\sum_{cyc} a}\right)^3\geq 64\prod_{cyc}(b+c)^2$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}\left(2s^2-4Rr-r^2+\frac{3r^2s}{s}\right)^3\geq 64.16R^2r^2s^2$$

$$\Leftrightarrow (s^2-2Rr+r^2)^3-128R^2r^2s^2\overset{(*)}{\geq}0 \text{ and } \because P=s^4(s^2-16Rr+5r^2)+$$
$$(10Rr-2r^2)s^2(s^2-16Rr+5r^2)+r^2(44R^2-94Rr+13r^2)(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0\left(\because 44R^2-94Rr+13r^2=(R-2r)(44R-6r)+r^2\overset{\text{Euler}}{\geq} r^2>0\right)$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\geq} P\Leftrightarrow$

$$(t-2)(87t^2-40t+4)\overset{?}{\geq}0\left(t=\frac{R}{r}\right)\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(*) \text{ is true}$$

$$\therefore \sqrt{2-(ab+bc+ca)+3abc}\geq 2\sqrt[3]{(a+b)(b+c)(c+a)}$$

$$\forall a,b,c>0\,|\,a+b+c=1,''='' \text{ iff } a=b=c=\frac{1}{3} \text{ (QED)}$$