ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then prove that :

$$\frac{\sqrt{bc}}{a+\sqrt{(a+b)(a+c)}}+\frac{\sqrt{ca}}{b+\sqrt{(b+c)(b+a)}}+\frac{\sqrt{ab}}{c+\sqrt{(c+a)(c+b)}}\geq 1$$

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$$\frac{\sqrt{bc}}{a+\sqrt{(a+b)(a+c)}}+\frac{\sqrt{ca}}{b+\sqrt{(b+c)(b+a)}}+\frac{\sqrt{ab}}{c+\sqrt{(c+a)(c+b)}}$$

$$\overset{CBS}{\geq}\sum_{cyc}\frac{\sqrt{bc}}{\sqrt{a+a+b}.\sqrt{a+a+c}}=\sum_{cyc}\frac{(bc)^{\frac{3}{2}}}{\sqrt{bc(2ac+bc)(2ab+bc)}}$$

$$\overset{Radon}{\geq}\frac{\left(\sum_{cyc}ab\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}\left(bc(2ac+bc)(2ab+bc)\right)}}\overset{?}{\geq}1$$

$$\Leftrightarrow\left(\sum_{cyc}ab\right)^{3}\overset{?}{\geq}\sum_{cyc}\left(bc(2ac+bc)(2ab+bc)\right)$$

$$\Leftrightarrow abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2-6abc\right)\overset{?}{\geq}0\rightarrow true\because\sum_{cyc}a^2b,\sum_{cyc}ab^2\overset{AM-GM}{\geq}3abc$$

$$\therefore\frac{\sqrt{bc}}{a+\sqrt{(a+b)(a+c)}}+\frac{\sqrt{ca}}{b+\sqrt{(b+c)(b+a)}}+\frac{\sqrt{ab}}{c+\sqrt{(c+a)(c+b)}}\geq 1$$

$\forall a,b,c>0,''='' \text{ iff } a=b=c$ (QED)