ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\geq 0$ and $2(ab+bc+ca)+abc=32$ then prove that :

$$ab+bc+ca\leq 2(a+b+c)$$

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$$2\sum_{cyc}((b+4)(c+4))-\prod_{cyc}(a+4)=32-abc-2\sum_{cyc}ab=0$$

$$\Rightarrow 2\sum_{cyc}((b+4)(c+4))=\prod_{cyc}(a+4)\Rightarrow\sum_{cyc}\frac{1}{a+4}=\frac{1}{2}\rightarrow(1)\ and\ \because\frac{1}{a+4}\overset{a\geq 0}{\leq}\frac{1}{4}$$

$$\therefore we\ can\ set:\frac{1}{a+4}=\frac{1}{4}-x,\frac{1}{b+4}=\frac{1}{4}-y,\frac{1}{c+4}=\frac{1}{4}-z\ (x,y,z\geq 0)$$

$$\Rightarrow x+y+z=\frac{3}{4}-\sum_{cyc}\frac{1}{a+4}\overset{via\ (1)}{=}\frac{1}{4}\therefore\sum_{cyc}x=\frac{1}{4}\rightarrow(2)\ and\ \because\frac{1}{a+4}=\frac{1-4x}{4}$$

$$\therefore a=\frac{4}{1-4x}-4=\frac{4x}{\frac{1}{4}-x}\overset{via\ (2)}{=}\frac{4x}{y+z}\ and\ so.\ a=\frac{4x}{y+z},b=\frac{4y}{z+x},c=\frac{4z}{x+y}$$

$$and\ hence,\ ab+bc+ca\leq 2(a+b+c)\ becomes:\sum_{cyc}\frac{16xy}{(y+z)(z+x)}\leq\sum_{cyc}\frac{8x}{y+z}$$

$$\Leftrightarrow\sum_{cyc}\left(x\left(x^2+\sum_{cyc}xy\right)\right)\geq 2\sum_{cyc}(xy(x+y))$$

$$\Leftrightarrow\sum_{cyc}x^3+\sum_{cyc}x^2y+\sum_{cyc}xy^2+3xyz\geq 2\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\geq\sum_{cyc}x^2y+\sum_{cyc}xy^2\rightarrow true\ via\ Schur$$

$$\therefore ab+bc+ca\leq 2(a+b+c)\ \forall\ a,b,c\geq 0\ |\ 2(ab+bc+ca)+abc=32,$$
$$''='' \ iff\ (a=b=c=2)\ or\ (a=0,b=c=4)\ or\ (b=0,c=a=4)$$
$$or\ (c=0,a=b=4)\ (QED)$$