ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)}\geq 4$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius R and inradius $=s,R,r$ (say); and then : $\sum_{cyc} a=s\rightarrow(1), abc=r^2s\rightarrow(2)$,

$$\sum_{cyc} ab=4Rr+r^2\rightarrow(3), \sum_{cyc} a^2=s^2-8Rr-2r^2\rightarrow(4)$$

$$\text{Now, } \frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)}=\sum_{cyc}\frac{(a^2+abc)^2}{a^2b^2+2abc^2}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a^2+3abc\right)^2}{\sum_{cyc} a^2b^2+2abc\sum_{cyc} a}\overset{a+b+c=3}{=}\frac{\left(\sum_{cyc} a^2+\frac{9abc}{\sum_{cyc} a}\right)^2}{\left(\sum_{cyc} ab\right)^2}\overset{\text{via (1),(2),(3) and (4)}}{=}$$

$$\frac{(s^2-8Rr-2r^2+9r^2)^2}{(4Rr+r^2)^2}\overset{?}{\geq}4\Leftrightarrow s^2-8Rr+7r^2\overset{?}{\geq}8Rr+2r^2\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2$$

$\rightarrow$ true via Gerretsen $\therefore \frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)}\geq 4$

$\forall a,b,c>0 \,|\, a+b+c=3,''='' \text{ iff } a=b=c=1$ (QED)