ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\geq -2$ and $a+b+c=3$, then prove that :

$$a^2+b^2+c^2+abc\geq 4$$

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$$\text{Let } a+2=x, b+2=y, c+2=z \text{ and then : } a^2+b^2+c^2+abc-4$$

$$=\sum_{cyc}(x-2)^2+\prod_{cyc}(x-2)-4$$

$$=\sum_{cyc}x^2-4\sum_{cyc}x+12+xyz-2\sum_{cyc}xy+4\sum_{cyc}x-8-4 \overset{x+y+z=9}{=}$$

$$\frac{1}{9}\left(\sum_{cyc}x^2-2\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)+xyz=\frac{1}{9}\left(\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\right)$$

$$\overset{Schur}{\geq} 0\ (\because x,y,z\geq 0)\ \therefore a^2+b^2+c^2+abc\geq 4\forall\ a,b,c\geq -2\ \Big|\ a+b+c=3,$$

$$''='' \text{ iff } (a=b=c=1) \text{ or } \left(a=-2,b=c=\frac{5}{2}\right) \text{ or } \left(b=-2,c=a=\frac{5}{2}\right)$$

$$\text{or } \left(c=-2,a=b=\frac{5}{2}\right)\ (QED)$$