ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ and $ab\le 1$, then prove that :

$$\frac{1}{a^3+b}+\frac{1}{b^3+a}\le\frac{1}{a+b}+\frac{4}{(a+b)^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3+b}+\frac{1}{b^3+a}=\frac{a^3+b^3+a+b}{a^3b^3+ab+a^4+b^4}\overset{A-G}{\le}\frac{a^3+b^3+a+b}{2a^2b^2+a^4+b^4}$$
$$=\frac{(a+b)^3-3ab(a+b)+a+b}{(a^2+b^2)^2}=\frac{x^3-3xy+x}{(x^2-2y)^2}\ (x=a+b \text{ and } y=ab)$$
$$\overset{?}{\le}\frac{1}{a+b}+\frac{4}{(a+b)^3}=\frac{4+x^2}{x^3}\Leftrightarrow(3-y)x^4+(4y^2-16y)x^2+16y^2\ \boxed{\overset{?}{\ge}0}\ (*)$$

and $\because$ LHS of $(*)$ is a quadratic polynomial in x^2 with discriminant $\delta=(4y^2-16y)^2-64y^2(3-y)=16y^2(2-y)^2\ \therefore$ in order to prove $(*)$,

it suffices to prove : $x^2\overset{?}{\ge}\frac{16y-4y^2+4y(2-y)}{2(3-y)}=\frac{24y-8y^2}{2(3-y)}$

$(\because (3-y),(2-y)>0 \text{ as } y\le 1)\ \to$ true $\because 2x^2(3-y)\overset{A-G}{\ge}8y(3-y)=24y-8y^2$

$\Rightarrow (*)$ is true $\therefore \frac{1}{a^3+b}+\frac{1}{b^3+a}\le\frac{1}{a+b}+\frac{4}{(a+b)^3}\ \forall\ a,b>0\ |\ ab\le 1,$

$''='' $ iff $a=b=1$ (QED)