ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>1$ and $2(a+b+c)=ab+bc+ca$, then prove that :

$$\frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}}\geq 2\sqrt{3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}}=\sum_{cyc}\frac{\sqrt{3}.a}{\sqrt{(b+1)(3b-3)}}\overset{A-G}{\geq}\sum_{cyc}\frac{\sqrt{3}.a}{\frac{b+1+3b-3}{2}}$$

$$=\sqrt{3}.\sum_{cyc}\frac{a}{2b-1}=\sqrt{3}.\sum_{cyc}\frac{a^2}{2ab-a}\overset{Bergstrom}{\geq}\frac{\sqrt{3}.\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab-\sum_{cyc}a}\overset{2(a+b+c)=ab+bc+ca}{=}$$

$$\frac{\sqrt{3}.\left(\sum_{cyc}a\right)^2}{4\sum_{cyc}a-\sum_{cyc}a}=\frac{\sum_{cyc}a}{\sqrt{3}}\geq\frac{6}{\sqrt{3}}\left(\because 2\sum_{cyc}a=\sum_{cyc}ab\leq\frac{\left(\sum_{cyc}a\right)^2}{3}\Rightarrow\sum_{cyc}a\geq 6\right)$$

$$\therefore\frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}}\geq 2\sqrt{3}$$

$$\forall\, a,b,c>1\,\big|\,2(a+b+c)=ab+bc+ca,''='' \text{ iff } a=b=c=2\ (\text{QED})$$