ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\in[0,2]$ and $a+b+c=3$, then prove that :

$$a^4+b^4+c^4\le 17$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Either $b\le 2, c<2$ or $c\le 2, b<2\therefore (b-2)(c-2)\ge 0$

$$\overset{a+b+c=3}{\Rightarrow} bc\ge 2(3-a)-4\Rightarrow bc\overset{(1)}{\ge} 2-2a$$

$$\text{Now, } a^4+b^4+c^4\le 17\Leftrightarrow a^4+(b+c)^4-2bc(2(b+c)^2-bc)\le 17$$

$$\overset{a+b+c=3}{\Leftrightarrow} 2b^2c^2-4bc(3-a)^2+a^4+(3-a)^4-17\le 0$$

$$\Leftrightarrow bc\overset{(m)}{\le}(3-a)^2+\sqrt{-6a^3+27a^2-54a+49} \text{ and}$$

$$bc\overset{(n)}{\ge}(3-a)^2-\sqrt{-6a^3+27a^2-54a+49}$$

$$\left(\because 16(3-a)^4-8(a^4+(3-a)^4-17)=16(-6a^3+27a^2-54a+49)\right.$$
$$\left.=16\left((2-a)(6a^2-15a+24)+1\right)>0\therefore \Delta_{6a^2-15a+24}<0 \text{ and } 2\ge a\right)$$

$$\text{Now, } bc\overset{A-G}{\le}\frac{(3-a)^2}{4}<(3-a)^2+\sqrt{-6a^3+27a^2-54a+49}$$

$\Rightarrow$ (m) is true and to prove (n), we segregate into 2 cases :

Case 1 $a\le 1$ and then : $bc\overset{via\,(1)}{\ge} 2-2a\overset{?}{\ge}$

$$(3-a)^2-\sqrt{-6a^3+27a^2-54a+49}$$

$$\Leftrightarrow \sqrt{-6a^3+27a^2-54a+49}\overset{?}{\ge}a^2-4a+7\Leftrightarrow$$

$$-6a^3+27a^2-54a+49\overset{?}{\ge}(a^2-4a+7)^2\left(\because \Delta_{a^2-4a+7}<0\Rightarrow a^2-4a+7>0\right)$$

$$\Leftrightarrow a(1-a)\left((1-a)+1+a^2\right)\overset{?}{\ge}0\to \text{true}\therefore a\le 1\Rightarrow (n) \text{ is true}$$

Case 2 $a>1$ and then : $bc\ge 0\overset{?}{\ge}(3-a)^2-\sqrt{-6a^3+27a^2-54a+49}$

$$\Leftrightarrow -6a^3+27a^2-54a+49\overset{?}{\ge}(3-a)^4\Leftrightarrow (a-1)(2-a)(a^2-3a+16)\overset{?}{\ge}0$$

$\to$ true $\because \Delta_{a^2-3a+16}<0\Rightarrow a^2-3a+16>0$ and $1<a\le 2\Rightarrow$ (n) is true

$\therefore$ combining both cases, (n) and (m) are true $\forall a,b,c\in[0,2]\,|\,a+b+c=3$

$$\therefore a^4+b^4+c^4\le 17\ \forall a,b,c\in[0,2]\,|\,a+b+c=3$$

"$=$" iff $(a=0,b=1,c=2)$ or $(a=0,b=2,c=1)$ or $(b=0,c=1,a=2)$ or $(b=0,c=2,a=1)$ or $(c=0,a=1,b=2)$ or $(c=0,a=2,b=1)$(QED)