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If $x,y,z>0$ and $xyz=1$, then prove that :

$$\frac{yz}{\sqrt{x+y}}+\frac{xz}{\sqrt{y+z}}+\frac{xy}{\sqrt{z+x}}\geq\frac{3\sqrt{2}}{2}$$

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$$\frac{yz}{\sqrt{x+y}}+\frac{xz}{\sqrt{y+z}}+\frac{xy}{\sqrt{z+x}}=\sum_{cyc}\frac{(xz)^{\frac{3}{2}}}{\sqrt{xyz+z^2x}}\overset{\text{Radon}}{\geq}$$

$$\frac{\left(\sum_{cyc}xy\right)^{\frac{3}{2}}}{\sqrt{3xyz+\sum_{cyc}x^2y}}\overset{\text{CBS}}{\geq}\frac{\left(\sum_{cyc}xy\right)^{\frac{3}{2}}}{\sqrt{3xyz+\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)}}}\overset{?}{\geq}\frac{3\sqrt{2}}{2}$$

$$\overset{xyz=1}{\Leftrightarrow}2\left(\sum_{cyc}xy\right)^3-27x^2y^2z^2\overset{?}{\geq}9xyz.\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)}$$

$$\Leftrightarrow\left(2\left(\sum_{cyc}xy\right)^3-27x^2y^2z^2\right)^2\overset{(*)}{\overset{?}{\geq}}81x^2y^2z^2\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0$,
$b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b$
$\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius
$=s,R,r$ (say) $\Rightarrow\sum_{cyc}x=s, xyz=r^2s, \sum_{cyc}xy=4Rr+r^2, \sum_{cyc}x^2=s^2-8Rr-2r^2$

and $\sum_{cyc}x^2y^2=r^2((4R+r)^2-2s^2)$ and so, $(*)$

$$\Leftrightarrow(2(4Rr+r^2)^3-27r^4s^2)^2-81r^4s^2.r^2((4R+r)^2-2s^2)(s^2-8Rr-2r^2)\overset{(**)}{\overset{?}{\geq}}0$$

and $\because P=162r^6(s^2-16Rr+5r^2)(s^2-4R^2-4Rr-3r^2)^2$
$+2r^6(972Rr+243r^2)(s^2-16Rr+5r^2)^2$
$-4r^6(648R^4+5616R^3r-11016R^2r^2+4050Rr^3+351r^4)\left(\begin{matrix}s^2-4R^2-4Rr\\-3r^2\end{matrix}\right)$

$\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\overset{?}{\geq}P$

$$\Leftrightarrow376t^6-2136t^5+10248t^4-23170t^3+17448t^2-1317t-1478\overset{?}{\geq}0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left((t-2)\left(\begin{matrix}376t^3(t-2)+120t^3+6216t^2\\+4222t+9472\end{matrix}\right)+19683\right)\overset{?}{\geq}0$$

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$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{is true} \therefore \frac{yz}{\sqrt{x+y}} + \frac{xz}{\sqrt{y+z}} + \frac{xy}{\sqrt{z+x}} \geq \frac{3\sqrt{2}}{2}$$

$$\forall x, y, z > 0 \,\big|\, xyz = 1, '' = '' \text{ iff } x = y = z = 1 \text{ (QED)}$$