ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z\geq 0$ and $x+y+z=3$, then prove that :

$$5\left(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}\right)+9\geq 8(xy+yz+zx)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Firstly $\forall\, x\geq 0$, we have : $5\sqrt[5]{x}+3x\geq 8\sqrt{x}\rightarrow$ (i) and $\because$ LHS $=$ RHS $=0$ for $x=0$, so we now focus on : $x>0$ and then : LHS of (i) $=$

$$\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+x+x+x\overset{A-G}{\geq}8\sqrt[8]{x.x^3}=8\sqrt{x}\Rightarrow \text{(i) is true}$$

$\forall\, x\geq 0$ and analogously, $5\sqrt[5]{y}+3y\geq 8\sqrt{y}$ and $5\sqrt[5]{z}+3z\geq 8\sqrt{z}$

$$\therefore 5\left(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}\right)+9\overset{x+y+z=3}{=}5\left(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}\right)+3(x+y+z)$$

$$\geq 8\sum_{cyc}\sqrt{x}\ \text{and so, it suffices to prove :}\ \sum_{cyc}a\overset{(*)}{\geq}\sum_{cyc}a^2b^2\ \text{subject to}\ \sum_{cyc}a^2=3$$

$\left(a=\sqrt{x},b=\sqrt{y},c=\sqrt{z}\right)$ and $(*)$ is trivially true when exactly 2 variables equal to zero and we now focus on the case when exactly 1 variable equals to zero and WLOG we may assume $a=0\ \left(b,c>0\,\middle|\,b^2+c^2=3\right)$ and then : $(*)\Leftrightarrow$

$$b+c\geq b^2c^2\overset{b^2+c^2=3}{\Leftrightarrow}(b^2+c^2).\sqrt{b^2+c^2}.(b+c)\geq b^2c^2.3\sqrt{3}\rightarrow\text{true}$$

(strict inequality) $\because$ LHS $\overset{A-G}{\geq}4\sqrt{2}.b^2c^2>3\sqrt{3}.b^2c^2\Rightarrow(*)$ is true and we now consider the case when : $a,b,c>0$ and assigning $b+c=X, c+a=Y, a+b=Z$

$$\Rightarrow X+Y-Z=2c>0, Y+Z-X=2a>0 \text{ and } Z+X-Y=2b>0$$

$$\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z \text{ form sides of a triangle XYZ}$$

with semiperimeter, circumradius & inradius $=s,R,r$ (say); and then :

$$\sum_{cyc}a=s\rightarrow(1),\ \sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(2)\ \text{and}\ \sum_{cyc}a^2b^2=$$

$$r^2((4R+r)^2-2s^2)\rightarrow(3)\ \text{and}\ (1),(2),(3)\ \text{together with}\ \sum_{cyc}a^2=3\Rightarrow(*)\Leftrightarrow$$

$$(s^2-8Rr-2r^2).\sqrt{s^2-8Rr-2r^2}.s\geq 3\sqrt{3}.r^2((4R+r)^2-2s^2)$$

$$\Leftrightarrow s^2(s^2-8Rr-2r^2)^3-27r^4((4R+r)^2-2s^2)^2\overset{(**)}{\geq}0\ \text{and}\ \because P=$$

$$(s^2-16Rr+5r^2)^4+2r(20R-13r)(s^2-16Rr+5r^2)^3$$

$$+2r^2(288R^2-396Rr+72r^2)(s^2-16Rr+5r^2)^2$$

$$+2r^3(1792R^3-2976R^2r+1308Rr^2+55r^3)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**)\overset{?}{\geq}P$$

$$\Leftrightarrow 160t^4-416t^3+60t^2+361t-194\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(160t^2+224t+316)+729\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\Rightarrow (*) \text{ is true } \textit{and} \text{ combining } \textit{all cases}, (*) \text{ is true } \forall\, a, b, c \geq 0 \,\Big|\, \sum_{cyc} a^2 = 3$$

$$\therefore 5\left(\sqrt[5]{x} + \sqrt[5]{y} + \sqrt[5]{z}\right) + 9 \geq 8(xy + yz + zx)\ \forall\, x, y, z \geq 0 \,\big|\, x + y + z = 3,$$
$$'' = ''\ \text{iff } x = y = z = 1\ (\text{QED})$$