ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\in[0,1]$ and $a+b+c=2$, then prove that :

$$a^6+b^6+c^6+2abc\le 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$b^6+c^6=(b+c)^6-6bc(b^4+c^4)-b^2c^2(15b^2+15c^2+20bc)$$
$$=(b+c)^6-6bc\left((b+c)^4-2bc(2(b+c)^2-bc)\right)-b^2c^2(15(b+c)^2-10bc)$$
$$\overset{a+b+c=2}{=}(2-a)^6-6bc(2-a)^4+9b^2c^2(2-a)^2-2b^3c^3$$
$$\le(2-a)^6-6bc(2-a)^4+9b^2c^2(2-a)^2-2b^2c^2(1-a)$$
$$\left(\because(b-1)(c-1)\ge 0\overset{a+b+c=2}{\Rightarrow}bc\ge 2-a-1\right)\Rightarrow a^6+b^6+c^6+2abc-2\boxed{\le}E$$
$$=(9(2-a)^2-2+2a)b^2c^2+(2a-6(2-a)^4)bc+a^6+(2-a)^6-2\to(1)$$

Now, E is a quadratic polynomial in bc with discriminant $\delta=$
$(2a-6(2-a)^4)^2-4(9(2-a)^2-2+2a)(a^6+(2-a)^6-2)$ and $\because P=$
$-36a(a-1)^7-124(a-1)^7-144a(a-1)^5+24a(a-1)^4+408(a-1)^4$
$-144(a-1)^3-116a(a-1)\ge 0\therefore$ in order to prove $\delta>0$,
it suffices to prove : $\delta>P\Leftrightarrow 108-92a>0\to$ true $\because a\le 1\therefore\delta>0$ and so,
in order to prove : $E\le 0$, it suffices to prove :

$$bc\boxed{\overset{(i)}{\le}}\frac{6(2-a)^4-2a+\sqrt{\delta}}{2(9(2-a)^2-2+2a)}\text{ AND }bc\boxed{\overset{(ii)}{\ge}}\frac{6(2-a)^4-2a-\sqrt{\delta}}{2(9(2-a)^2-2+2a)}$$

$\because bc\ge 1-a$ and $\because 9(2-a)^2-2+2a\overset{a\le 1}{\ge}7>0\therefore$ in order to prove (ii),
it suffices to prove : $2(1-a)(9(2-a)^2-2+2a)\overset{?}{\ge}6(2-a)^4-2a-\sqrt{\delta}$

$$\Leftrightarrow\sqrt{\delta}\overset{?}{\ge}2(a-2)(3a^3-9a^2+11a-7)$$
$$\Leftrightarrow(2a-6(2-a)^4)^2-4(9(2-a)^2-2+2a)(a^6+(2-a)^6-2)$$
$$\overset{?}{\ge}4(a-2)^2(3a^3-9a^2+11a-7)^2$$
$$\left(\because 3a^3-9a^2+11a-7=(a-1)(3(a-1)^2+2)-2\overset{a\le 1}{\le}-2<0\right.$$
$$\left.\overset{a\le 1<2}{\Rightarrow}(a-2)(3a^3-9a^2+11a-7)>0\right)$$
$$\Leftrightarrow 4a(1-a)\left(18a^6-104a^5+303a^4-591a^3+716a^2-442a+136\right)\overset{?}{\ge}0$$
$$\Leftrightarrow 18a^6-104a^5+303a^4-591a^3+716a^2-442a+136\boxed{\overset{?}{\ge}}0\text{ and }\because$$

$Q=18(a-1)^6+4a(a-1)^4+49(a-1)^4-59(a-1)^3\ge 0\therefore$ in order
to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}Q\Leftrightarrow 9(1-a^2)+1+35a\overset{?}{\ge}0$
$\to$ true(strict inequality) $\Rightarrow(*)\Rightarrow$ (ii) is true and again, via AM − GM,

$$bc\le\frac{(2-a)^2}{4}\overset{?}{\le}\frac{6(2-a)^4-2a+\sqrt{\delta}}{2(9(2-a)^2-2+2a)}$$
$$\Leftrightarrow 24(2-a)^4-8a-2(9(2-a)^2-2+2a)(2-a)^2+4\sqrt{\delta}\overset{?}{\ge}0$$

$\Leftrightarrow 3a^4-26a^3+82a^2-116a+56+2\sqrt{\delta}\overset{?}{\geq}0$ $(**)$ and it's trivially true when :

$3a^4-26a^3+82a^2-116a+56\geq 0$ and so, we now focus on the case when : $3a^4-26a^3+82a^2-116a+56<0$ and then : $(**)\Leftrightarrow$

$$2\sqrt{\delta}\overset{?}{\geq}-(3a^4-26a^3+82a^2-116a+56)\Leftrightarrow$$

$$4(2a-6(2-a)^4)^2-16(9(2-a)^2-2+2a)(a^6+(2-a)^6-2)$$

$$\overset{?}{\geq}(3a^4-26a^3+82a^2-116a+56)^2$$

$$\Leftrightarrow a\left((a-1)^4.\Big((1-a)(153a^2+97a+251)+745\Big)+(a+186)(a-1)^2\right.$$

$$\left.+63+931(1-a)\right)\overset{?}{\geq}0$$

$\rightarrow$ true $\therefore 0\leq a\leq 1\Rightarrow (**)\Rightarrow$ (i) is true and so, both (i) and (ii) are true $\Rightarrow E\leq 0$

$\overset{\text{via }(1)}{\Rightarrow}$ $a^6+b^6+c^6+2abc-2\leq 0\therefore a^6+b^6+c^6+2abc\leq 2\ \forall\ a,b,c\in[0,1]$ such that $a+b+c=2$, "$=$" iff $(a=0,b=1,c=1)$ or $(b=0,c=1,a=1)$ or $(c=0,a=1,b=1)$ (QED)